8.2 Mixed Stackelberg and Cournot Oligopoly

$$X = X_R + X_G = \sum_{i=1}^{m} x_i + \sum_{i=m+1}^{n} x_i$$

$$P(X) = b\,(K - X) + c$$

$$\max_{x_i} \quad \pi_R = P(X)x_i - cx_i$$

$$\begin{aligned}
\frac{d\pi_R}{dx_i} = P(X) + P'(X)x_i - c &\overset{!}{=} 0 \\
P(X) + P'(X)x_i &= c \\
P(X) + P'(X)\frac{1}{m}X_R &= c \\
b\,(K - X) + c - b\,\frac{1}{m}X_R &= c \\
b\,(K - X_R - X_G) - b\,\frac{1}{m}X_R &= 0 \\
K - X_G - \left(1 + \frac{1}{m}\right) X_R &= 0 \\
X_R &= \frac{1}{1 + \frac{1}{m}}\,(K - X_G)
\end{aligned}$$

$$\max_{X_G} \quad \pi_G = P(X)X_G - cX_G$$

$$\begin{aligned}
\pi_G &= P(X)X_G - cX_G \\
&= (b\,(K - X) + c)\,X_G - cX_G \\
&= b\,(K - X_R - X_G) + cX_G - cX_G \\
&= b\left(K - \frac{1}{1 + \frac{1}{m}}\,(K - X_G) - X_G\right) X_G \\
&= b\left(\left(1 - \frac{1}{1 + \frac{1}{m}}\right) K - \left(1 - \frac{1}{1 + \frac{1}{m}}\right) X_G\right) X_G \\
&= bX_G\,(K - X_G)\,\frac{1}{1 + m}
\end{aligned}$$

$$\begin{aligned}
\frac{d\pi_G}{dX_G} = b\,(K - X_G)\,\frac{1}{1 + m} - bX_G\frac{1}{1 + m} &\overset{!}{=} 0 \\
K - X_G - X_G &= 0 \\
X_G &= \frac{1}{2}K
\end{aligned}$$

Mixed Stackelberg and Cournot Oligopoly (continued)

$$\begin{aligned} X_A &= \frac{1}{\frac{1}{n}+1}K \\ &= \frac{n}{1+n}K \end{aligned}$$

$$\begin{aligned} X_S &= X_R + X_G \\ &= \frac{1}{\frac{1}{m}+1}\left(K - \frac{1}{2}K\right) + \frac{1}{2}K \\ &= \frac{m}{1+m}\frac{1}{2}K + \frac{1}{2}K \\ &= \frac{1}{2}K\left(\frac{m}{1+m}+1\right) \\ &= \frac{1}{2}K\frac{1+2m}{1+m} \end{aligned}$$

$$\begin{aligned} X_S &> X_A \\ \frac{1}{2}K\frac{1+2m}{1+m} &> \frac{n}{1+n}K \\ (1+2m)(1+n) &> 2n(1+m) \\ 1+2m+n+2mn &> 2n+2mn \\ 1+m &> n-m \end{aligned}$$